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Commutative BCK-Algebras and Quantum Structures[†]

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We study commutative BCK-algebras with the relative cancellation property, i.e., if $a \leq x, y$ and $x * a = y * a$, then $x = y$. Such algebras generalize Boolean rings as well as Boolean D-posets (= MV-algebras). We show that any such BCK-algebra X can be embedded into the positive cone of an Abelian lattice-ordered group. Moreover, this group can be chosen to be a universal group for X. We compare BCK-algebras with the relative cancellation property with known quantum structures as posets with difference, D-posets, orthoalgebras, and quantum MV-algebras, and we show that in many cases we obtain MV-algebras.

1. INTRODUCTION

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [ImIs]. This notion originated from two different avenues; (1) set theory and (2) classical and nonclassical propositional calculi. BCK-algebras have been studied by many authors and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory, and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (= MV-algebras).

Investigation of the mathematical foundations of quantum mechanics had shown that the Kolmogorov model of probability theory holding for classical mechanics fails in the case of quantum mechanics. Birkhoff and von Neumann [BiNe] introduced a *quantum logic*, i.e., an algebraic system describing a propositional system of quantum mechanics. This system is more general than Boolean algebras, and the most important example of a quantum logic is the quantum logic $\mathscr{L}(\mathrm{H})$ consisting of all closed subspaces

[†] This paper is dedicated to the memory of Prof. G. T. Rüttimann.

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of a real, complex, or quaternionic Hilbert space H. More general structures are *orthoalgebras* introduced by Foulis and Randall [FoRa]. Recently Kôpka and Chovanec [KoCh], students of the present author, introduced *difference posets* (D-posets in abbreviation), which combine both algebraic and fuzzy sets ideas. The most important examples are the interval [0, 1] and the system $\mathscr{E}(H)$ of all effect operators, i.e., of all Hermitian operators A of a real, complex, or quaternionic Hilbert spaces H such that $O \leq A \leq I$, where I is the identity. These operators play a key role in the so-called unsharp approach to quantum mechanics [BLM], when orthogonal projections (having the spectrum in $\{0, 1\}$) are "sharp" analogues of events in quantum mechanics, while other effect operators (having the spectrum in [0, 1]) are "unsharp" or "fuzzy" analogues of ones.

An equivalent structure is an *effect algebra* introduced originally by Giuntini and Greuling [GiGr] as a weak orthoalgebra, and Foulis and Bennett [FoBe].

The primary notion of D-posets is a *difference*, which has some similar properties to the difference $*$ defined in BCK-algebras. Therefore, in the present paper we shall investigate connections between posets with difference and BCK-algebras, and among D-posets and BCK algebras, MV-algebras introduced by Chang [Cha1] (arising from multivalued logic) and quantum MV-algebras presented by Giuntini [Giu], and orthoalgebras, respectively. We show that in many cases they coincide with BCK-algebras only for commutative BCK-algebras, respectively only for Boolean D-posets ($=$ MV-algebras) or Boolean algebras.

On the other hand, we show that for any MV-algebra, its radical, which gives important information on the propositional system, is always a commutative BCK-algebra not bounded.

We recall that MV-algebras have a very close connection with C*-algebra, more precisely with approximately finite-dimensional (for short, AF) C*-algebras, because according to [Mun2, Theorem 4.2], every countable MV-algebra is in a one-to-one correspondence with AF C*-algebras applied to Elliott's classification (with a so-called Murray–von Neumann order on the set of projections).

On the other hand, we stress that in some physically very important examples which are only D-posets and not a BCK-algebra, like the set of effect operators on H, $\mathscr{E}(H)$, there are sub D-posets which have a BCK-structure (in the mentioned case, e.g., the set of all constants). In addition, the problem of compatibility in D-posets will lead to using BCK-algebra or MV-algebra structures in sub D-posets. On the other hand, an important consequence of applying the theory of BCK-algebras to quantum structure is the ℓ-group structure of a given propositional (sub)system. ℓ-Groups or po-groups are a very important and traditional part of mathematics which it is now possible to apply to quantum structures as was underlined by Greechie and Foulis [GrFo]. Therefore, the commutative BCK-algebras and MV-algebras can also be useful for quantum structure theory.

2. BCK-ALGEBRAS

According to Imai and Iséki [ImIs], a BCK-algebra is defined as follows:

Definition 2.1. A *BCK-algebra* $(X; *, 0)$ is a nonempty set X with a binary operation $*$ and with a constant element 0 such that the following axioms are satisfied. For all $x, y, z \in X$:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$.
(BCK-2) $(x * (x * y)) * y = 0$.
(BCK-3) $x * x = 0$.
(BCK-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.
(BCK-5) $0 * x = 0$.

The partial ordering $\leq$ on X is defined by

$$x \leq y \quad \text{iff} \quad x * y = 0$$

and then the BCK-algebra is a poset X with a fixed element 0 and with a binary operation $*$ satisfying the following axioms:

(BCK-1′) $(x * y) * (x * z) \leq z * y$.
(BCK-2′) $x * (x * y) \leq y$.
(BCK-3′) $x \leq x$.
(BCK-4′) $x \leq y$ and $y \leq x$ imply $x = y$.
(BCK-5′) $0 \leq x$.

A BCK-algebra X is said to be *commutative* if

$$x * (x * y) = y * (y * x), \qquad x, y \in X$$

then

$$x \wedge y = x * (x * y), \qquad x, y \in X$$

and any commutative BCK-algebra is a $\wedge$-semilattice.

X is *bounded* if there exists $1 \in X$ such that $x * 1 = 0$, $x \in X$, i.e., $x \leq 1$; in this case we will write $(X; *, 0, 1)$. Every Boolean algebra (we define $a * b := a \wedge b'$), every MV-algebra (see Theorem 2.4) is a bounded commutative BCK-algebra. For example, $([0, \infty); *_R, 0)$, where

$$s *_R t = \max\{0, s - t\}$$

$s, t \in [0, \infty)$, is an unbounded commutative BCK-algebra, and $([0, 1]; *_R,$

0, 1) is a bounded commutative BCK-algebra, in fact, an MV-algebra, and ([0, 1); $*_R$, 0) is an unbounded commutative BCK-algebra. In addition, any bounded commutative BCK-algebra is a distributive lattice.

A poset (L; $\ominus$, $\leq$, 0) with a partial order $\leq$, a least element 0, and a partial binary operation $\ominus$, called a *difference* on L, such that $b \ominus a$ is defined iff $a \leq b$, is said to be a *poset with difference* [KoCh] if the followings axioms are satisfied. For all a, b, $c \in L$:

(DPi) $b \ominus a \leq a$.
(DPii) $b \ominus (b \ominus a) = a$.
(DPiii) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

If (L; $\ominus$, $\leq$, 0) has the greatest element 1, it is said to be a *D-poset*.

Example 2.2. Let H be a real, complex, or quaternionic Hilbert space. Denote by $\mathscr{E}(H)$ the set of all effect operators on H, i.e., of all Hermitian operators A on H such that $O \leq A \leq I$, where O and I are the null and identity operators on H. The partial order on $\mathscr{E}(H)$ is defined via $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$. The partial operation $\ominus$ is defined via $B \ominus A = B - A$ iff $A \leq B$, where $-$ is the usual subtraction of operators. Then ($\mathscr{E}(H)$; $\ominus$, O, I) is a D-poset which is not a lattice and no commutative BCK-algebra, in which the BCK-order coincides with the original partial order of effect operators.

For the proof of the following result see [DvKi].

Theorem 2.3 Let (X; $*$, 0) be a BCK-algebra. We define a partial binary operation $\div$ on X such that, for x, $y \in X$, $y \div x$ is defined if and only if $x \leq y$, and in this case

$$y \div x := y * x$$

Then:

(i) $y \div x \leq y$ if $x \leq y$.
(ii) $y \div (y \div x) \leq x$ if $x \leq y$.
(iii) If $x \leq y \leq z$, then $z \div y \leq z \div x$ and $(z \div x) \div (z \div y) \leq y \div x$.

The partial binary operation $\div$ is a difference on (X; $\leq$) if and only if (X; $*$, 0) is a commutative BCK-algebra.

Many-valued analogues of a two-valued logic are MV-algebras introduced by Chang [Cha1], and, according to Mundici [Mun1], they can be characterized as follows. An MV-*algebra* is a nonempty set L with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : L \times L \to L$, and with a unary operation *: $L \to L$ such that, for all $a,b,c \in L$, we have:

(MVi) $a \oplus b = b \oplus a$ (commutativity).
(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity).
(MViii) $a \oplus 0 = a$.
(MViv) $a \oplus 1 = 1$.
(MVv) $(a^*)^* = a$.
(MVvi) $a \oplus a^* = 1$.
(MVvii) $0^* = 1$.
(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We define the binary operations $\odot$, $\vee$, $\wedge$ as follows:

$$a \odot b := (a^* \oplus b^*)^*, \qquad a, b \in L$$

$$a \vee b := (a^* \oplus b)^* \oplus b, \qquad a, b \in L$$

$$a \wedge b := (a^* \vee b^*)^*, \qquad a, b \in L$$

Then $(L; \odot, 1)$ is a semigroup written "multiplicatively" with the neutral element 1.[2] If, for $a, b \in L$, we define

$$a \leq b \Leftrightarrow a = a \wedge b$$

then $\leq$ is a partial order on L, and $(L; \vee, \wedge, 0, 1)$ is a distributive lattice with the least and greatest elements 0 and 1, respectively [Cha1]. We recall that $a \leq b$ iff $b \oplus a^* = 1$.

The following results have been proved in [DvKi].

Theorem 2.4. Let $(X; * 0, 1)$ be a bounded BCK-algebra with the induced order $\leq$. We define:

(i) A partial binary operation $\div$ defined via (2.1).

(ii) A binary operation $\oslash$ on X defined via[3]

$$y \oslash x := y \div (y * (y * x)), \qquad x,y \in X$$

(iii) A unary operation *: $X \to X$ defined via

$$x^* := 1 \div x, \qquad x \in X$$

(iv) Binary operations $\oplus$ and $\odot$ on X defined via

$$x \oplus y := (x^* \oslash y)^*, \qquad x, y \in X$$

$$x \odot y := x \oslash y^*, \qquad x, y \in X$$

Then, for all $x, y \in X$,

[2] We remark that in the literature, by an MV-algebra is assumed the structure $(L; \oplus, \odot, ^*, 0, 1)$, where $\odot$ is defined as above.

[3] We recall that $y * (y * x)$ and $x * (x * y)$ are lower bounds of x and y.

$$y \oslash x = y * x, \qquad x \oplus y = (x^* * y)^*, \qquad x \odot y = (x^* \oplus y^*)^*$$

and the following statements are equivalent:

(a) $(X; \leq, \div, 0, 1)$ is a D-poset.
(b) $(X; *, 0, 1)$ is a bounded commutative BCK-algebra.
(c) $(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra.

If this is the case, then the orders determined by the BCK-algebra and the MV-algebra coincide.

Theorem 2.5. Let a poset with difference $(X; \leq, \ominus)$ be a lower semilattice with respect to $\leq$. Define a binary operation $*$ on X by

$$x * y := x \ominus (x \wedge y), \qquad x, y \in X \tag{2.2}$$

Then

$$x * y = x \ominus y \Leftrightarrow y \leq x, \qquad x, y \in X$$

and X possesses a least element 0. Moreover, $(X; *, 0)$ is a commutative BCK-algebra if and only if, for all $x, y, z \in X$, we have

$$(x * y) * z = (x * z) * y \tag{2.3}$$

It is necessary to mention that property (2.3) holds in any BCK-algebra (not only in commutative ones). Therefore, starting with posets being lower semilattices with difference, they induce via (2.2) a BCK-algebra iff (2.3) holds; in such a case, the BCK-algebra is commutative. For example, the difference in $\mathcal{P}(H)$ does not entail BCK-structure compatible with the original difference. In addition, in orthomodular lattices such a condition obtains given an OML is a Boolean algebra.

It is worth recalling that even if a given structure is a D-poset which is not a commutative BCK-algebra with a binary operation $*$ compatible with the difference, sometimes it is possible to find a sub D-poset which has such a property. For example, in $\mathcal{E}(H)$, the set of all constants has such a property, i.e., (2.3) holds here. In addition, the problem of compatibility on lattice D-posets can lead to condition (2.3) on a sub D-poset.

On the other hand, of $(X; \leq, 0)$ is a poset, then $(X; *, 0)$, where $x * y = x$ if $x \not\leq y$ and $x * y = 0$ if $x \leq y$, gives a noncommutative BCK-algebra such that the BCK order and the original one coincide.

We recall that Chovanec and Kôpka [ChKo] introduced an important family of D-posets called *Boolean D-posets*; similarly Bennett and Foulis [BeFo] introduced *Phi symmetric effect algebras*. It is worth to recalling that all these structures appeared in natural and independent way in quantum structures, but all are equivalent to MV-algebras.

Giuntini [Giu] recently introduced an interesting class of algebraic structures, a quantum many-valued algebra (*QMV-algebra*), which generalizes MV-algebras. Axiom (MVviii) is responsible for the lattice-theoretic behavior of the corresponding operations $\vee$ and $\wedge$ defined via $a \vee b = (a^* \oplus b)^* \oplus b$ and $a \wedge b = (a \oplus b^*) \odot b$ for any $a, b \in L$.

Theorem 2.6. Let $(X; \oplus, \odot, {}^*, 0, 1)$ be a QMV-algebra and define a binary operation* on X via

$$x * y := x \odot y^*, \qquad x, y \in X$$

Then $(X; *, 0)$ is a BCK-algebra if and only if $(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra. If this is the case, then $(X; *, 0, 1)$ is a bounded commutative BCK-algebra.

The converse statement holds, too, starting with a bounded BCK-algebra.

Theorem 2.7. Let an orthoalgebra $(X; \oplus, 0, 1)$ be a lower semilattice. Define a binary operation $*$ on X via

$$x * y = (x^{\perp} \oplus (x \wedge y))^{\perp}, \qquad x, y \in X$$

Then the following statements are equivalent:

(i) $(x * y) * z = (x * z) * y$, $x, y, z \in X$
(ii) $(X; *, 0, 1)$ is a bounded implicative BCK-algebra.[4]
(iii) $(X; \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra with $x \oplus y = x \vee y$ for all $x; y \in X$.
(iv) $(X; \leq, \vee, \wedge, {}^*, 0, 1)$ is a Boolean algebra.

In addition, the converse implication holds, too.

3. COMMUTATIVE BCK-ALGEBRAS WITH THE RELATIVE CANCELLATION PROPERTY

A commutative BCK-algebra $(X; *, 0)$ has the *relative cancellation property* if, for $a, x, y \in X$, $a \leq x, y$ with $x * a = y * a$ implies $x = y$.

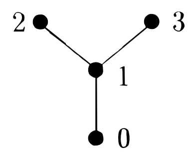
Every upward-directed or bounded commutative BCK-algebra has the relative cancellation property.

Example 3.1. $(\{0, 1, 2, 3\}; *, 0)$ is a commutative BCK-algebra which is not directed upward, consequently it is not a lattice. It does not have the

[4] A BCK-algebra $(X; *, 0)$ is called *implicative* if, for all $x, y \in X$ we have $x = x * (y * x)$.

relative cancellation property, because $1 \leq 2, 3$ and $2 * 1 = 1 = 3 * 1$, but $2 \neq 3$,

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	1	1	0



Consequently, X cannot be embedded into the positive cone of an Abelian l-group.

We define a partial binary operation $+$ via $a + b = c$ iff $c \geq a$ and $c * a = b$. Here, $+$ is commutative, associative, and cancellative, and 0 is a neutral element.

Example 3.2. Suppose that $(G; +, \leq, 0)$ is an Abelian l-group with the positive cone $G^+ = \{g \in G: g \geq 0\}$. Then $(G^+; *_G, 0)$ is a commutative BCK-algebra with the relative cancellation property, where $*_G$ is defined via

$$u *_G v := (u - v) \vee 0$$

for $u, v \in G^+$. More generally, if G_0 is a nonvoid subset of G^+ such that u, $v \in G_0$ implies $u *_G v \in G_0$, then $(G_0; *_G, 0)$ is a commutative BCK-subalgebra of $(G^+; *_G, 0)$ having the relative cancellation property.

Theorem 3.3. Let $(X; *, 0)$ be a commutative BCK-algebra with the relative cancellation property. Then there exists an Abelian l-group $(G; +$ $\leq, 0)$ with the positive cone G^+ and a nonvoid subset G_0 of G^+ generating G^+ such that $u, v \in G_0$ implies $u *_G y \in G_0$, and there exists a BCK-isomorphism h from X onto G_0.

In addition, X has a *universal group*, i.e., a pair (G, h), where G is an Abelian l-group and $h: X \to G$ such that (i) $h(X)$ generates G^+, (ii) h preserves $+$, and (iii) for any partially ordered Abelian group G_1 and any order and $+$-preserving mapping $g: X \to G_1$ there is a group-homomorphism of ordered groups $g': G \to G_1$ such that $g = g' \circ h$.

We denote by $\mathcal{BCK}$ the *category* whose objects are commutative BCK-algebras and morphisms are BCK-homomorphisms.

Let G_1 and G_2 be two Abelian l-groups. A mapping $h: G_1 \to G_2$ is said to be an *l-group homomorphism* iff h is both a group-homomorphism and a lattice-homomorphism; in other words, for each $a, b \in G$, $h(a + b) = h(a) + h(b)$, $h(a \wedge b) = h(a) \wedge h(b)$ (as well as for joins).

We denote by $\mathcal{LG}$ the *category* whose objects are pairs (G, G_0), where G is an Abelian l-group and G_0 is a nonvoid subset of the positive cone G^+ of G such that G_0 generates G^+ and $(G_0; *_G, 0)$ is a BCK-algebra.

A *morphism* from (G, G_0) into (G', G'_0) is an l-group homomorphism h: $G \to G'$ such that $h(G_0) \subseteq G'_0$.

Let now (G, G_0) be an object of $\mathcal{LG}$ and define a morphism $\mathcal{X}$ from the category $\mathcal{LG}$ into the category $\mathcal{BCK}$ as follows;

$$\mathcal{X}(G, G_0) = (G_0; *_G, 0)$$

Theorem 3.4. $\mathcal{X}$ is a faithful and full functor from the category $\mathcal{LG}$ of Abelian l-groups into the category $\mathcal{BCK}$ of commutative BCK-algebras with the relative cancellation property. In addition, $\mathcal{X}$ defines a categorical equivalence.

For example, let (G, u) be a unital Abelian l-group with a strong unit u, and define $G_0(u) = \{g \in G: 0 \leq g \leq u\}$, with operations

$$a \oplus_G b := u \wedge (a + b)$$

$$a \odot_G b := 0 \vee (a + b - u)$$

$$a^{*_G} := u - a$$

We obtain an MV-algebra and Mundici's [Mun1] famous categorical representation of MV-algebras via unital l-groups. We recall that our representation is in frames of so-called interval effect algebras, when an effect algebra, or equivalently a D-poset appears as a unit interval in some partially ordered Abelian group. Such a connection of quantum structures in the newest fashion with po-groups gives us an unexpected application of standard part of mathematics, po-groups, to quantum structure theory, as was underlined by Greechie and Foulis [GrFo].

4. MEASURES ON COMMUTATIVE BCK-ALGEBRAS

Let $(X; *, 0)$ be a BCK-algebra (not necessarily commutative). A mapping m: $X \to [0, \infty)$ is said to be (i) a *measure* if $m(x * y) = m(x) - m(y)$ whenever $y \leq x$; (ii) a *measure-morphism* if $m(x * y) = m(x) *_R m(y)$, $x, y \in X$. If, in addition, $1 \in X$ and $m(1) = 1$, m is said to be a *state* or a *state-morphism*.

$I \subseteq X$ is an *ideal* of a BCK-algebra X if:

- (i) $0 \in I$.
- (ii) If $x * y \in I$ and $y \in I$, then $x \in I$.

If we define $x \sim_I y$ iff $x * y \in I$ and $y * x \in I$, then $\sim_I$ defines a congruence, and X/I is a BCK-algebra, too.

It is interesting to recall that if m is a measure on X, then $I_m := \{x \in X: m(x) = 0\}$ is an ideal of X, and the quotient X/I_m is always a commutative BCK-algebra.

We denote by $\mathcal{M}(X)$ the set of all maximal ideals. It is worth to recall that in contrast to MV-algebras (= Boolean D-posets, or equivalently to bounded commutative BCK-algebras), it can be empty.

Define recursively, for all $x, y \in X$,

$$x *^0 y = x, \qquad x *^1 y = x * y, \ldots, x *^{n+1} y = (x *^n y) * y, \qquad n \geq 1$$

An elemenlt $u \in X$ is a *quasi strong unit* for X iff given x there exists an integer $n \geq 1$ such that $x *^n u = 0$. This is equivalent to $h(u)$ being a strong unit for (G, h).

If X has a quasi strong unit, then $\mathcal{M}(X) \neq \emptyset$.

A *radical*, $Rad(X)$, of a commutative BCK-algebra $(X; *, 0)$ is defined by

$$Rad(X) = \cap\{I: I \in \mathcal{M}(X)\}$$

X is *semisimple* if $Rad(X) = \{0\}$.

We now show how unbounded commutative BCK-algebras can appear in many MV-algebras studying radicals.

If X is a bounded commutative BCK-algebra (= an MV-algebra), then the radical $R(X)$ exists and $(Rad\ (X); *\ 0)$ is not necessarily a bounded commutative BCK-algebra in which $a + b \in Rad(X)$ for all $a, b \in Rad(X)$.

The radical of a commutative BCK-algebra or of a Boolean D-poset gives important information concerning the corresponding quantum structure, as follows from the following two results.

Theorem 4.1. Let u be a quasi strong unit of a nontrivial commutative BCK-algebra $(X; *, 0)$ with the relative cancellation property. Let $S_u(X)$ be the set of all measures m on X such that $m(u) = 1$. Then $S_u(X)$ is a nonempty compact convex Hausdorff space, and the space of all measure-morphisms from $S_u(X)$ is a nonvoid compact Hausdorff space. Any measure from $S_u(X)$ is a weak limit of the convex hull of the set of extremal points of $S_u(X)$. In addition, the following statements are equivalent:

(i) m is an extremal measure from $S_u(X)$.
(ii) m is a measure-morphism from $S_u(X)$.
(iii) $m(x \wedge y) = \min\{m(x), m(y)\}$, $x, y \in X$, $m(u) = 1$.

Theorem 4.2. Let u be a quasi strong unit of a nontrivial commutative BCK-algebra $(X; *, 0)$ with the relative cancellation property. The following statements are equivalent:

(i) X is semisimple.
(ii) X is Archimedean.
(iii) X has an order-determining system of measure-morphisms from $S_u(X)$.

(iv) X has an order-determining system of measures from $S_u(X)$.
(v) X is isomorphic to some commutative BCK-algebra of functions on some $\Omega \neq \emptyset$.[5]

According to the above, if m is a measure on X, then

$$I_m := \{x \in X: m(x) = 0\}$$

is an ideal of X, and there exists a one-to-one correspondence between the set of all measure morphisms and maximal ideals, respectively, given by the correspondence $m \mapsto I_m$.

It is worth recalling that the notion of a state on MV-algebras defined, e.g., by [Mun2] or on bounded commutative BCK-algebras coincide.

5. CONCLUDING REMARKS

In the paper, we have studied commutative BCK-algebras and their connection to the newest quantum structures like D-posets, MV-algebra, quantum MV-algebras (QMV-algebras), as well as orthoalgebras. Quantum structures are various kinds of algebraic structures motivated by mathematical foundations of quantum mechanics.

We have given necessary and sufficient conditions for a BCK-algebra and for a D-poset to be the same as an MV-algebra (Theorems 2.3–2.5).

We have proved that a quantum MV-algebra can be reformulated as a BCK-algebra iff it is an MV-algebra (Theorems 3.3 and 3.4).

We have stressed that among unbounded BCK-algebras can be found many BCK-algebras of the form $(Rad(X), *, 0)$, where X is an MV-algebra.

In addition, we have described categorical equivalence of the category of commutative BCK-algebras with the relative cancellation property with a special category of l-groups, Theorems 3.3 and 3.4, and described the state space of commutative BCK-algebras.

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REFERENCES

[BeFo] M. K. Bennett and D. J. Foulis, A generalized Sasaki projection for effect algebras, *Tatra Mt. Math. Publ.* **15** (1998) 55–66.

[5] I.e., a system of functions $\mathcal{F} \subseteq [0, \infty)^{\Omega}$ such that (i) $0_{\Omega} \in \mathcal{F}$ and (ii) if $f, g \in \mathcal{F}$, then $f * g \in \mathcal{F}$, where $(f * g)(\omega) := \max\{f(\omega) - g(\omega), 0\}$, $\omega \in \Omega$.

[BiNe] G. Birkhoff and J. von Neumann, The logic of quantum mechanics, *Ann. Math.* **37** (1936) 823–834.

[BLM] P. Busch, P. J. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement*, Springer-Verlag, Berlin (1991).

[Cha1] C. C. Chang, Algebraic analysis of many valued logics, *Trans. Am. Math. Soc.* **88** (1958) 467–490

[Cha2] C. C. Chang, A new proof of the completeness of the Łukasiewicz axioms, *Trans. Am. Math. Soc.* **93** (1959) 74–80.

[ChKo] F. Chovanec and F. Kôpka, Boolean D-posets, *Tatra Mt. Math. Publ.* **10** (1997) 183–197.

[Dvu1] A. Dvurečenskij, On categorical equivalence of BCK-algebras, *Studia Logica* **64** (2000), 21–36.

[Dvu2] A. Dvurečenskij, Commutative BCK-algebras and lattice ordered groups with universal property, *Indian J. Math*, submitted.

[DvGr] A. Dvurečenskij and M. G. Graziano, Commutative BCK-algebras and lattice ordered groups, *Math. Japonica* **49** (1999) 159–174.

[DvKi] A. Dvurečenskij and H. S. Kim, Connections between BCK-algebras and difference posets, *Studia Logica* **60** (1998) 421–439.

[FoBe] D. Foulis and M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.* **24** (1994) 1331–1352.

[FoRa] D. J. Foulis and C. H. Randall, Operational statistics. I. Basic concepts, *J. Math. Phys.* **13** (1972) 1667–1675.

[GiGr] R. Giuntini and H. Greuling, Toward a formal language for unsharp properties, *Found. Phys.* **19** (1989) 931–945.

[Giu] R. Giuntini, Quantum MV algebras, *Studia Logica* **56** (1996) 393–417.

[GrFo] R. J. Greechie and D. J. Foulis, Transition to effect algebras, *Int. J. Theor. Phys.* **34** (1995) 1369–1382.

[ImIs] Y. Imai and K. Iséki, On axiom systems of propositional calculi, *Proc. Japan Acad.* **42** (1966) 19–22.

[KoCh] F. Kôpka and F. Chovanec, D-posets, *Math. Slovaca* **44** (1994) 21–34.

[Mun1] D. Mundici, Interpretation of AF C*-algebras in Łukasiewicz sentential calculus, *J. Funct. Anal.* **65** (1986) 15–63.

[Mun2] D. Mundici, Averaging the truth-value in Łukasiewicz logic, *Studia Logica* **55** (1995) 113–127.